ON CERTAIN CONSTRAINTS WITH FRICTION

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The problem of the motion of mechanical systems constrained by frictional connections is more than of practical interest only. Usually, such systems are reduced to systems with smooth connections by incorporating the forces of friction with the given forces; nevertheless, a direct application of the method of Lagrange permits the establishment of general principles for such systems without any explicit introduction of the constraint reactions.

1. We consider a mechanical system of n points with masses m_i, which have the coordinates x_i, y_i, z_i relative to certain fixed orthogonal axes. Let the system be constrained by certain linear connections.

The possible displacements of the points m_i under the imposed constraints, and at a fixed moment of time t, we denote by δx_i, δy_i, δz_i.

From the possible displacements we isolate all displacements which satisfy the conditions

$$x_i'\delta x_i + y_i'\delta y_i + z_i'\delta z_i = 0 \qquad (i = 1,\ldots,n) \tag{1.1}$$

where x_i', y_i', z_i' denote the actual velocities of the points m_i at the chosen moment t. We shall call these the S-displacements.

We suppose that given forces X_i, Y_i, Z_i act on the points m_i. The differential equations of motion of the constrained material system are

$$m_i x_i'' = X_i + R_{x_i}, \quad m_i y_i'' = Y_i + R_{y_i}, \quad m z_i'' = Z_i + R_{z_i} \qquad (i = 1,\ldots,n) \tag{1.2}$$

where R_{x_i}, R_{y_i}, R_{z_i} denote the reaction forces imposed on the system by the connections.

The most usual frictional connections are determined by the axiom

$$\sum (R_{x_i}\delta x_i + R_{y_i}\delta y_i + R_{z_i}\delta z_i) = 0 \tag{1.3}$$

which holds for any S-displacements δx_i, δy_i, δz_i.

The frictional-connections axiom (1.3) postulates that the work of the reactions acting on the material system m_i at the chosen instant of time t, when the real velocities of the points m_i are x_i', y_i', z_i', is zero for any arbitrary S-displacements.

If we eliminate the reactions R_{xi}, R_{yi}, R_{zi} from the axiom (1.3) in accordance with the differential equations (1.2) of the real motion, then we obtain the following relations for the real motions:

$$\sum[(m_i x_i'' - X_i)\,\delta x_i + (m_i y_i'' - Y_i)\,\delta y_i + (m_i z_i'' - Z_i)\,\delta z_i] = 0 \qquad (1.4)$$

which holds for any S-displacements δx_i, δy_i, δz_i.

It is of interest to note that the constraint reactions R_{xi}, R_{yi}, R_{zi} do not enter into the relation (1.4), which plays a principal role. Actually, upon multiplying relations (1.1) by an undetermined multiplier μ_i and adding it to (1.4) we obtain

$$\sum[(m_i x_i'' - X_i - \mu_i x_i')\,\delta x_i + (m_i y_i'' - Y_i - \mu_i y_i')\,\delta y_i + \\ + (m_i z_i'' - Z_i - \mu_i z')\,\delta z_i] = 0 \qquad (1.5)$$

This equality holds for any possible displacements δx_i, δy_i, δz_i if the multiplier μ_i was so chosen that $\mu_i x_i'$ (or $\mu_i y_i'$ or $\mu_i z_i'$) is equal to the corresponding projection of frictional force (with x_i', y_i', z_i' not zero). But under these conditions the last expression represents a known principle in the dynamics of material systems constrained by frictional connections.

2. In order to establish the differential equations of a material system from the principle enunciated in (1.5), we suppose that the imposed constraints are expressed by the general relations

$$\sum (a_i^{(s)}\delta x_i + b_i^{(s)}\delta y_i + c_i^{(s)}\delta z_i) = 0 \qquad (s = 1, \ldots, m) \qquad (2.1)$$

Upon multiplying these constraint equations (2.1) by an undetermined multiplier λ_s and as an additional restriction, the S-displacements by μ_i, and combining with (1.4, we have

$$\sum[(m_i x_i'' - X_i - \sum_s \lambda_s a_i^{(s)} - \mu_i x_i')\,\delta x_i + (m_i y_i'' - Y_i - \sum_s \lambda_s b_i^{(s)} - \mu_i y_i')\,\delta y_i + \\ + (m_i z_i'' + Z_i - \sum_s \lambda_s c_i^{(s)} - \mu_i x_i')\,\delta z_i] = 0 \qquad (2.2)$$

By choosing $n + m$ of the multipliers so that the coefficients in the last expression vanish for $n + m$ dependent S-displacements x_i, y_i, z_i, we obtain for such a choice of μ_i and λ_s only terms in the last expression with $2n - m$ independent displacements δx_i, δy_i, δz_i; the coefficients for the independent displacements must be zero and consequently we have

$$\begin{aligned} m_i x_i'' &= X_i + \sum \lambda_s a_i^{(s)} + \mu_i x_i' \\ m_i y_i'' &= Y_i + \sum \lambda_s b_i^{(s)} + \mu_i y_i' \qquad (i = 1,\ldots,n) \qquad (2.3) \\ m_i z_i'' &= Z_i + \sum \lambda_s c_i^{(s)} + \mu_i z_i' \end{aligned}$$

For determination of the multipliers λ_s, the expressions for the material system constraints are used, written in the form of a relation for admissible velocities as a refinement of (2.1):

$$\sum (a_i^{(s)} x_i' + b_i^{(s)} y_i' + c_i^{(s)} z_i') + e^{(s)} = 0 \qquad (s = 1,\ldots,m) \qquad (2.4)$$

The relations for determining the multipliers μ_i, being insufficient, must be either a refinement of relations (1.1) or else the values of the multipliers μ_i must be given beforehand as characteristics of the frictional forces.

3. If the geometrical relations (2.4) are integrated, then it is possible to employ the Lagrangian method to eliminate the multipliers λ_s; it follows that the geometrical relations must be expressed by means of new holonomic variables $q_1, \ldots, q_k$

$$x_i = x_i(q_1,\ldots,q_k,t), \quad y_i = y_i(q_1,\ldots,q_k,t), \quad z_i = z_i(q_1,\ldots,q_k,t)$$
$$(i = 1,\ldots,n; \quad k = 3n - m) \qquad (3.1)$$

From this the possible displacements are obtained as the relations

$$\delta x_i = \sum \frac{\partial x_i}{\partial q_s} \delta q_s, \qquad \delta y_i = \sum \frac{\partial y_i}{\partial q_s} \delta q_s, \qquad \delta z_i = \sum \frac{\partial z_i}{\partial q_s} \delta q_s \qquad (3.2)$$

Upon substitution of these values into (1.5) we have

$$\sum \delta q_s \left[\frac{d}{dt} \frac{\partial T}{\partial q_s'} - \frac{\partial T}{\partial q_s} - Q_s + \frac{\partial f}{\partial q_s'} \right] = 0 \qquad (3.3)$$

if the μ_i do not depend on the velocities and where f denotes the dissipation function

$$f = -\frac{1}{2} \sum \mu_i (x_i'^2 + y_i'^2 + z_i'^2) = f_2 + f_1 + f_0 \qquad (3.4)$$

From this we obtain the equations of motion in the form

$$\frac{d}{dt} \frac{\partial T}{\partial q_s'} - \frac{\partial T}{\partial q_s} = Q_s - \frac{\partial f}{\partial q_s'} \qquad (s = 1,\ldots,k) \qquad (3.5)$$

in which the multipliers μ_i, still undetermined, have entered into the dissipation function f. For given values of the μ_i multipliers, the dissipation function f is completely determined, and consequently it is possible to express the characteristics of the frictional connections by means of a dissipation function, if the μ_i depend only on the time and position of the system.

4. Upon multiplying Equations (3.5) by q_s' and adding, we shall have

$$\frac{d}{dt}(T_2 - T_0) = \sum Q_s q_s' - 2f_2 - f_1 \tag{4.1}$$

Therefore $2f_2 + f_1$ determines the rate of dissipation of mechanical energy.

Usually, in frictional connections, the mechanical energy is dissipated as heat; for such connections the quantity $2f_2 + f_1$ will be related to the heat developed by friction.

5. For definiteness it must be noted that the so-called dry friction, reduced by Coulomb [1] to unilateral smooth constraints is not included in this paper.

In connection with the theory of dry friction it may be noted that a point of mass m moving under applied forces X, Y with unilateral constraints

$$y \geqslant -a\cos\frac{x}{b} \tag{5.1}$$

where a and b are extremely small quantities, has the following differential equation of motion for $(\lambda > 0)$:

$$mx'' = -\lambda\frac{a}{b}\sin\frac{x}{b} + X, \quad my'' = \lambda + Y \tag{5.2}$$

The rough approximation $X + Y = 0$ gives the known result of Coulomb, that the value of the frictional force is

$$\begin{matrix}\max\\ \min\end{matrix}\left| -Y\frac{a}{b}\sin\frac{x}{b}\right|$$

which is proportional to Y and is independent of the velocity x'.

If in the exact expression for λ

$$\lambda = \frac{-mY + m\frac{a}{b^2}x'^2\cos\frac{x}{b} + X\frac{a}{b}\sin\frac{x}{b}}{m + \frac{a^2}{b^2}\sin^2\frac{x}{b}}$$

only the principal terms are retained, then the corresponding value of the frictional force

$$\left|\left(-Y + \frac{a}{b^2}x'^2\cos\frac{x}{b}\right)\frac{a}{b}\sin\frac{x}{b}\right|$$

will increase at first from the Coulomb value with the growth in x', while the value of x, after breaking away from the preceding crest to fall to a second crest, will lie to the left of the point of maximum frictional force; after this, the frictional force will decrease.

BIBLIOGRAPHY

1. Coulomb, *Théorie des machines simples*, 1785.

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